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Perturbative analysis of generalized Einstein's theories

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Abstract

The hypothesis that the energy-momentum tensor of ordinary matter is not conserved separately, leads to a non-adiabatic expansion and, in many cases, to an Universe older than usual. This may provide a solution for the entropy and age problems of the Standard Cosmological Model. We consider two different theories of this type, and we perform a perturbative analysis, leading to analytical expressions for the evolution of gravitational waves, rotational modes and density perturbations. One of these theories exhibits satisfactory properties at this level, while the other one should be discarded.

1 Introduction

The Cosmological Standard Model is based on the General Relativity theory, coupled to a perfect fluid energy-momentum tensor, with geometry determined by the Robertson-Walker metric. It predicts an initial hot phase, when the baryogenesis and primordial nucleosynthesis take place, followed by the matter dominated

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phase, when galactical structures are formed. This model can be considered as successful from many points of view, but it requires very stringent initial conditions. For example, the entropy observed today is of the order of 10^{87} [1], which can be also expressed by saying that the actual density is very close to the critical density necessary for the existence of a spatially flat Universe.

The thermodynamic equilibrium must also be imposed as an initial condition, at least while the nucleosynthesis takes place, due to the existence of the horizon distance smaller than the Hubble radius at that period. Moreover, the formation of structures are well explained if we admit the mean density fluctuations of the order $\Delta \propto 10^{-5}$ when the radiation decouples from matter; but there is no clear mechanism explaining the amplification up to this value of small initial perturbations, originating in the very beginning of the Universe.

The inflationary model, based on the existence of a de Sitter phase prior to the nucleosynthesis, copes with the entropy and horizon problem. It also proposes a mechanism of generation and evolution of inhomogeneties. In what concerns the explanation of structure formation, inflation is in competition with other mechanisms, like those proposed by topological defect models. In contrast to the latter, it proposes gaussian perturbations of quantum mechanical origin. Until very recently, the observational results of the COBE satellite concerning the anisotropies of the Cosmic Microwave Background, seemed to favorize strongly the inflationary mechanism, but the gaussian nature of the perturbations has been questioned short time ago [2]. If this result is confirmed, it will become a serious drawback for the inflationary models.

Various other cosmological models try to propose alternative solutions to the problems that remain open in the Standard Cosmological Model, without using the inflationary paradigm. Many of them introduce "exotic" types of matter sources, or modify the dynamical equations[3]. The models introducing a "cosmic string fluid" [4] can lead to good results concerning the horizon problem, but exhibit instabilities at perturbative level [5]. Another interesting possibility consists in considering a "non conservative" theory of gravity, meaning a gravity theory in which the energy-momentum tensor does not conserve in the usual way. The propositions of this type go back to the work of Rastall [6], and have again attracted some attention recently [7, 8]. Such a scheme leads to an older Universe in comparison with the standard results, creating at the same time more entropy during its evolution.

On a purely formal level, these "non-conservative" theories are equivalent to the introduction of exotic matter, whose main feature is negative pressure. More fundamentally, they lead to a radical modification of the dynamics of the Universe. We find frequently in the litterature statements explaining that such theories are incompatible with the equivalence principle. But the equivalence principle means that particles follow geodesic curves satisfying the equation $u^\mu u_{\nu;\mu} = 0$, and this requirement is totally independent of ${T^{\mu\nu}}_{;\mu} = 0$, and there is no incompatibility between the assumption ${T^{\mu\nu}}_{;\mu} \neq 0$ and the equivalence principle [6].

Here we propose to study these theories from the perturbative point of view,

paying special attention to the evolution of gravitational waves, rotational modes and density perturbations. The perturbative analysis intends to give more information concerning the structure formation in the realm of these theories, checking their stability at the same time.

In order to do this, we shall consider two different models. The first one, referred to as "Theory I", is based on the Brans-Dicke theory, rewritten in the Einstein frame. In this frame, matter is necessarily coupled to the scalar field. Certain considerations on the physical aspects of the two different frames have been presented in reference [9]. In spite of serious objections against the theories of this kind, we will consider it here mainly as a reference model. The second model, referred to as "Theory II", is based on the original proposition due to Rastall, and in our opinion has better physical justification. The results obtained in the framework of this model indicate that the improvements with respect to the standard scenario are not compromised, at least at the perturbative level. This theory must be also implemented with a convenient mechanism for generation and proper evolution of inhomogeneties in an expanding globally homogenous and isotropic Universe.

In the next section, we present briefly the two theories and discuss their cosmological solution and some of its implications. In section 3, we perform a perturbative study, considering perturbations of spin 2, 1 and 0. The comments and conclusions are contained in section 4.

2 Generalized Einstein's theories

The main point to be assured in order to solve the entropy problem without inflation, is to introduce a non adiabatic evolution of the Universe. This can be obtained by giving up the condition that the energy-momentum tensor has a vanishing covariant divergence. We can achieve this goal essentially in two different ways. First, we can start with a non-minimal coupling between the gravity and the scalar field on the one hand, and the matter on the other hand (like e.g. in the Brans-Dicke theory); then, through a conformal transformation we can recover the minimal coupling between the gravity and the scalar field, but with a more complicate coupling between matter, the metric field and the scalar field. The second possible choice is to modify Einstein's equations in such a way that the Bianchi identities are no longer satisfied in their normal form, with a conserved energy-momentum tensor on the right hand side, and in order to rectify the situation, we must modify the conservation laws for matter, which amounts to include new relationship between the gravitational and other forms of energy.

We will call the first choice *Theory I* and the second one *Theory II*. We shall consider now separately each of these possibilities, analyzing their implications on the corresponding cosmological scenarios.

2.1 Theory I

We start with the Brans-Dicke theory coupled to the matter in the ordinary way

$$L = \sqrt{-g}\left(\phi R - \omega \frac{\phi_{;\rho}\phi^{;\rho}}{\phi}\right) + L_m(g^{\mu\nu}) \quad . \tag{1}$$

Now, we redefine the field ϕ and perform a conformal transformation as in [10]

$$\begin{aligned} \phi &= \frac{1}{2\kappa} e^{-2\beta\sigma} \quad , & (2) \\ \tilde{g}_{\mu\nu} &= e^{2\beta\sigma} g_{\mu\nu} \quad , & (3) \\ \beta &= \frac{1}{2\sqrt{\omega + \frac{3}{2}}} \quad . & (4) \end{aligned}$$

Combining the equations (2, 3, 4), we obtain a new lagrangian which can be written as :

$$L = \frac{\sqrt{-g}}{\kappa}\left(R - \sigma_{;\rho}\sigma^{;\rho}\right) + L_m(e^{2\beta\sigma} g_{\mu\nu}) \quad , \tag{5}$$

where we have suppressed the tildes in order to simplify the notation. The variational principle leads then to the following equations :

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R &= \kappa T_{\mu\nu} + \sigma_{;\mu}\sigma_{;\nu} - \frac{1}{2} g_{\mu\nu}\sigma_{;\rho}\sigma^{;\rho} \quad , & (6) \\ \Box\sigma &= -\kappa\beta T \quad , & (7) \\ T^{\mu\nu}{}_{;\mu} &= \beta\sigma^{;\nu} T \quad , & (8) \end{aligned}$$

where $T = g^{\mu\nu} T_{\mu\nu}$ is the trace of the energy-momentum tensor. Imposing the Robertson-Walker metric

$$ds^2 = dt^2 - \frac{a^2(t)}{1 + \frac{k}{4}r^2}(dx^2 + dy^2 + dz^2) \quad , \tag{9}$$

and with the energy-momentum tensor of a perfect fluid, the above system reduces to the following three equations :

$$\begin{aligned} 3\frac{\dot{a}^2}{a^2} + 3\frac{k}{a^2} &= \kappa\rho + \frac{1}{2}(\dot{\sigma})^2 \quad , & (10) \\ \ddot{\sigma} + 3\frac{\dot{a}}{a}\dot{\sigma} &= -\kappa\beta(\rho - 3p) \quad , & (11) \\ \dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) &= \beta\dot{\sigma}(\rho - 3p) \quad . & (12) \end{aligned}$$

The solutions to these equations for the case $k = 0$ are easily obtained :

- $\alpha = -1$:

$$a \propto t^r \quad , \quad r = \frac{1}{8\beta^2} \quad , \quad \sigma = -\frac{1}{2\beta}\ln t \quad ; \tag{13}$$

- $\alpha = 0$:

$$a \propto t^r \quad , \quad r = \frac{2}{3+2\beta^2} \quad , \quad \sigma = -\frac{4\beta}{3+2\beta^2}\ln t \quad ; \tag{14}$$

- $\alpha = \frac{1}{3}$:

$$a \propto t^r \quad , \quad r = \frac{1}{2} \quad , \quad \sigma = constant \quad . \tag{15}$$

Note that when $-\frac{3}{2} < \omega < \infty$, $\infty > \beta^2 > 0$.

These solutions display the following characteristic features :

1. The solution for the radiative Universe is the same as in General Relativity, since the trace of the energy-momentum tensor is zero;
2. The solution for $\alpha = -1$ exhibits an inflationary behaviour when $\omega > \frac{1}{2}$. The same situation occurs in the original Brans-Dicke theory [11];
3. The solution for the material phase does not exhibit an inflationary behavior and the scale factor evolves always slower than in the standard model ($0 < r < \frac{2}{3}$). This implies that the age of the Universe in this case is less than in the standard case since $t_0 = rH_0$, where H_0 is Hubble's constant measured today.

Concerning the entropy production, using the equation (12), and the relation $S = \frac{(\rho+p)a^3}{T}$ [1], where S and T are the total entropy and temperature of the Universe, we obtain the following solution for S,

$$S = S_0 \exp\beta\sigma = S_0 \exp(-\frac{\beta^2}{3+2\beta^2}\ln t) \quad . \tag{16}$$

Hence the entropy always decreases with time during the material phase, while it remains constant during the radiative phase.

2.2 Theory II

The modified field equations are written with a a new parameter λ as

$$R_{\mu\nu} - \frac{\lambda}{2} g_{\mu\nu} R = \kappa T_{\mu\nu} \quad . \tag{17}$$

The divergence of (17) leads to

$$T^{\mu\nu}{}_{;\mu} = \frac{1-\lambda}{2\kappa} R^{;\nu} \quad . \tag{18}$$

Obviously these equations coincide with the standard ones when $\lambda = 1$. We can write this system of field equations in an equivalent form as

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa \left(T_{\mu\nu} - \frac{\gamma-1}{2} g_{\mu\nu} T \right) \quad , \tag{19}$$

$$T^{\mu\nu}{}_{;\mu} = \frac{\gamma-1}{2} T^{;\nu} \quad , \tag{20}$$

$$\text{where} \quad \lambda = \tfrac{2-\gamma}{3-2\gamma}, \quad \text{and} \qquad \gamma = \tfrac{3\lambda-2}{2\lambda-1}\,.$$

One checks easily that the covariant divergence of the energy-momentum tensor vanishes and the equations become standard when $\gamma = 1$ (i.e. with $\lambda = 1$). This form of the system suggests that the modification amounts in fact to a redefining the energy-momentum tensor by inclusion of part of gravitational effects directly, and does not affect the geometric framework of the theory.

This type of modification is less radical than the departure from the Einstein-Hilbert lagrangian consisting in replacement of the scalar curvature R by a function $f(R)$, as in [12]; the common feature remaining in both cases is that the extra geometrical terms may be interpreted as a modification of the energy-momentum tensor.

It is generally argued that theories of this type can not be derived from a variational principle. However, we can define a matter lagrangian leading to the equations (19, 20). For example the matter lagrangian of the form :

$$L_m = \sqrt{-g}\Big(x(\rho+p)u^\mu u^\nu g_{\mu\nu} + y\rho + zp\Big) \tag{21}$$

leads to Einstein's equations when $x = 1$, $y = -1$ and $z = -3$, and to the generalized Einstein's equations when we choose $x = 1$, $y = -\gamma$ and $z = -3\gamma$.

We look for solutions with spatially flat Robertson-Walker metric (9), where $a(t)$ is the scale factor describing the evolution of the Universe. We consider the energy-momentum tensor of a perfect fluid,

$$T^{\mu\nu} = (\rho+p)u^\mu u^\nu - pg^{\mu\nu} \quad , \tag{22}$$

where ρ, p and u^μ denote respectively, the energy density, the pressure and the four-velocity of the fluid. We assume a barotropic equation of state $p = \alpha\rho$, where α shall take on the values -1, $\frac{1}{3}$ and 0, corresponding to the vacuum energy density, radiation and matter dominated equation of state respectively, representing the most important cases for the primordial cosmological models.

For these three equations of state we can easily determine simple solutions, namely,

- $\alpha = -1$:

$$a(t) \;\propto\; \exp Ht \quad , \quad H = \sqrt{\frac{k}{3}(3-2\gamma\rho)} \quad , \tag{23}$$

$$\rho(t) \;\propto\; constant \quad ; \tag{24}$$

- $\alpha = \frac{1}{3}$:

$$a(t) \propto t^{\frac{1}{2}} \quad , \quad \rho(t) \propto t^{-2} \quad ; \tag{25}$$

- $\alpha = 0$:

$$a(t) \propto t^{1-\frac{\gamma}{3}} \quad , \quad \rho(t) \propto t^{-2} \quad ; \tag{26}$$

Among the above solutions only the case $\alpha = 0$ presents an improvement with respect to the standard model. The age of the Universe for this case can be evaluated as being equal to

$$t_0 = (1 - \frac{\gamma}{3}) H_0{}^{-1} \quad , \tag{27}$$

leading to a Universe older than in standard mocdel for $\gamma < 1$. On the other hand, using the equation (20), and the same relations between ρ, p, S and T as in the previous section, we can deduce the entropy growth in the matter dominated Universe according to the power law

$$S = S_0 t^{1-\gamma} \quad . \tag{28}$$

Hence, the condition $\gamma < 1$ implies at once an older Universe and an increase of the total entropy during the matter dominated phase. Moreover, the Universe exhibits an inflationary regime in the matter dominated era if $\gamma < 0$. We can also note that due to the particular coupling between matter and gravity, it is possible to get negative values of the critical mass density parameter Ω and at same time a closed or spatially flat Universe[7]. This feature is quite similar to the one displayed by a model which couples ordinary matter with a string induced fluid, when closed Universe can evolve dynamically as an open one [4]. However, it turns out that such models are hydrodynamically unstable [5].

3 Perturbative analysis

The Theory I considered above does not lead, in principle, to satisfactory results relative to the age of the Universe and the entropy generation. The Theory II, on the other hand, gives acceptable results if the parameter γ is restricted to the values $0 < \gamma < 1$. A perturbative analysis of these theories may give new insights into the physical consequences they lead to. In spite of previously shown negative results from the Theory I, we analyze it also from the perturbative point of view, for the sake of completeness. Our aim in both cases is to check if these theories can describe adequately the primordial gravitational waves, vectorial modes and density perturbations. In other words, we would like to see if they can lead to growing vectorial modes, associated with rotations, which is another drawback of the standard model. Also, if they can provide a mechanism for the amplification of density perturbations compatible with observations, since it is generally assumed that they are very small in the beginning of the Universe, and must be amplified during all its history in order to explain the observed large scale structures, as well as the observed distortions of the Cosmic Background Radiation.

Technically, we use the standard procedure of Lifschitz-Khalatnikov [13]. We isert into the field equations the perturbed quantities $\tilde{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$, $\tilde{\rho} = \rho + \delta\rho$ and $\tilde{p} = p + \delta p$, where $g_{\mu\nu}$, ρ and p are the background solutions around which small inhomogeneties are created. Using the invariance of field equations with respect

to the infinitesimal coordinate transformations, we may fix coordinate conditions eliminating unphysical degrees of freedom. Here we shall use the *synchronous coordinate condition* $h_{\mu 0} = 0$. It is well known [14], that after that a residual coordinate freedom remains, leading to a non-physical degree of freedom in the density perturbations. In our final results, we will consider the physical modes only. The derivation of the perturbed field equations is standard. In addition to the considerations exposed above, we note that the perturbed quantities h_{ij} can be split into a sum of tensorial, vectorial and scalar modes as follows:

$$h_{ij} = h^{TT}{}_{ij} + h^{T}{}_{(i|j)} + h^{s}_{1|i|j} + h^{s}_{2}\gamma_{ij} \quad , \tag{29}$$

where the first term is the transverse tracelless component (spin 2), the second one represents the vectorial mode (spin 1) and the last two ones are identified as scalar modes (spin 0). The derivatives are defined in the three-dimensional spatial section and γ_{ij} is the induced metric of this spatial section. Moreover, it can be shown that the spatial components of the perturbed functions obey the eigenvalue equation $\nabla^2 Q = -k^2 Q$, where $k^2 = \vec{k}^2$, and $\vec{k}$ is the wave vector of the perturbation which is supposed to propagate as a plane wave.

We shall consider now the perturbed equations and their solutions in each of the aformentioned theories. In all the following formulae, $k^2 = \vec{k}^2$.

3.1 Theory I

- Spin 2:

$$\ddot{h}_{ij} - \frac{\dot{a}}{a}\dot{h}_{ij} + \Big(\frac{k^2}{a^2} + 4\,\frac{\ddot{a}}{a} - (1-\alpha)[3\,\frac{\dot{a}^2}{a^2} - \frac{1}{2}\frac{\dot{\sigma}^2}{2}]\Big)h_{ij} = 0 \quad ; \tag{30}$$

- Spin 1:

$$\delta\dot{u}^i + \Big[(2-3\alpha)\,\frac{\dot{a}}{a} + \beta\dot{\sigma}\,(1-3\alpha)\Big]\,\delta u^i = 0 \quad ; \tag{31}$$

- Spin 0:

$$\begin{aligned}
\ddot{h} + 2\frac{\dot{a}}{a}\,\dot{h} - 4\dot{\sigma}\dot{\chi} - \Big(3\,(\frac{\dot{a}^2}{a^2}) - \frac{1}{2}\,\dot{\sigma}^2\Big)(1+3\alpha)\Delta &= 0 \quad ; &(32)\\
\ddot{\chi} + 3\frac{\dot{a}}{a}\,\dot{\chi} - \nabla^2\frac{\chi}{a^2} - \frac{1}{2}\dot{\sigma}\dot{h} + \Big(3\,(\frac{\dot{a}^2}{a^2}) - \frac{1}{2}\,\dot{\sigma}^2\Big)(1-3\alpha)\,\Delta &= 0 \quad ; &(33)\\
\dot{\Delta} + (1+\alpha)\Big(\Theta - \frac{1}{2}\,\dot{h}\Big) - \beta(1-3\alpha)\dot{\chi} &= 0 \quad ; &(34)\\
(1+\alpha)\dot{\Theta} + (1+\alpha)\Big((2-3\alpha)\frac{\dot{a}}{a} + \beta(1-3\alpha)\dot{\sigma}\Big)\Theta &+ \\
+\frac{\nabla^2}{a^2}\Big(\alpha\Delta + \beta(1-3\alpha)\,\chi\Big) &= 0 \quad . &(35)
\end{aligned}$$

In the expressions above, $\Delta = \frac{\delta\rho}{\rho}$, $\Theta = \delta u^i_{,i}$, $h = \frac{h_{kk}}{a^2}$ and $\chi = \delta\sigma$.

The equation describing the tensorial mode admits a simple solution when $\alpha = 0$ in terms of the conformal time defined as $dt = ad\eta$:

$$h^{TT} \propto \eta^{\frac{1+r}{2(1-r)}} c_\pm J_{\pm\nu}(k\eta) \tag{36}$$

where $\nu = \frac{3-2\beta^2}{2(1+2\beta^2)}$, $J_{\pm\nu}$ are Bessel functions with two possible values of index, $\pm\nu$, and $c_\pm$ are constants of integration. The results are essentially similar to those obtained in the General Relativity framework for $\alpha = \frac{1}{3}$ [15], and in the original Brans-Dicke theory [16] for $\alpha = -1$. The vectorial mode equation also has a simple solution with

$$\delta u^i \propto t^{\frac{4(\beta^2-1)}{3+2\beta^2}} \quad . \tag{37}$$

Growing rotational modes can appear if $\beta^2 > 1$. This leads to the restriction $-\frac{3}{2} < \omega < -\frac{5}{4}$, implying $\frac{1}{3} < r < \frac{2}{5}$. s It should be noted that such values of r correspond to a very young Universe.

For the density perturbations (spin 0), the solution is exactly the same as for the radiative phase in General Relativity. As it is well known, there is no significant amplification of the perturbations during this phase [15]. Nevertheless, unlike in General Relativity, we can have non-zero solutions for the perturbations during the de Sitter phase. Choosing $\alpha = -1$ in the perturbed equations, we find first that $\Delta = 4\beta\chi$. Using this relation, we can determine a third order diferential equation satisfied by Δ:

$$\begin{aligned} \dddot{\Delta} + \Big(5\frac{\dot a}{a} - \frac{\ddot\sigma}{\dot\sigma}\Big)\ddot\Delta + \\ +\Big[(\frac{k^2}{a^2}) + 3\Big(\frac{\ddot a}{a} + (\frac{\dot a^2}{a^2}) - \frac{\dot a}{a}\frac{\ddot\sigma}{\dot\sigma}\Big) - 2\dot\sigma^2 + 16\beta^2\Big(3(\frac{\dot a^2}{a^2}) - \frac{1}{2}\dot\sigma^2\Big)\dot\Delta + \\ +\Big[-\frac{\ddot\sigma}{\dot\sigma}(\frac{k}{a})^2 + 16\beta^2\Big(3\frac{\dot a}{a}(2\frac{\ddot a}{a} - \frac{\ddot\sigma}{\dot\sigma}\frac{\dot a}{a}) - \frac{\dot a}{a}\dot\sigma^2 - \frac{1}{2}\dot\sigma\ddot\sigma\Big) \\ +4\beta\dot\sigma\Big(3(\frac{\dot a}{a})^2 - \frac{1}{2}\dot\sigma^2\Big)\Big]\Delta = 0 \end{aligned} \tag{38}$$

As usual, $\Delta \propto \frac{1}{t}$ is a solution that is related to the residual coordinate freedom. Inserting $\Delta = \frac{1}{t}\zeta$ and using the conformal time such that $dt = ad\eta$, then writing $\zeta = \eta^p\xi$, where $p = \frac{3}{2}$, we find the following equation for ξ:

$$\xi'' + \frac{\xi'}{\eta} + \Big[1 - \frac{1}{4}(\frac{1+8\beta^2}{1-8\beta^2})^2\frac{1}{\eta^2}\Big]\xi = 0 \quad . \tag{39}$$

This is a Bessel equation. Turning back to Δ, we have the solution:

$$\Delta = \frac{1}{\eta^{s+1}}\int \eta^{\frac{3}{2}} c_\pm J_{\pm\nu}(k\eta)d\eta \quad , \tag{40}$$

with $\nu = \frac{1}{2}\frac{1+8\beta^2}{1-\beta^2}$ and $s = \frac{1}{8\beta^2-1}$.

This solution is not valid for $n = 1$ when this is the case, we find an Euler equation,:

$$\dddot{\Delta} + 6\frac{\ddot{\Delta}}{t} + (k^2 + 6)\frac{\dot{\Delta}}{t^2} + k^2\Delta = 0 \quad . \tag{41}$$

Discarding the solution connected with the residual coordinate freedom we have :

$$\Delta = t^{-1\pm\sqrt{1-k^2}} \quad . \tag{42}$$

The main feature of these solution is that asymptotically, for small k (large wavelength of the perturbations), it does not contain growing modes.

For $\alpha = 0$, we can find solutions in the long wavelength limit, $k \to 0$. They read :

$$\Delta \propto t^p \quad , \quad p = \frac{1}{2(3+2\beta^2)}\biggl(-1 + 6\beta^2 \pm (5 - 2\beta^2)\biggr) \quad . \tag{43}$$

The maximum value of p appears when $\omega \to \infty$, leading to $p = \frac{2}{3}$. In this limit the theory coincides with General Relativity. Hence, during the material phase, the perturbations grow less rapidly than in the standard model.

3.2 Theory II

For this theory, the perturbed equations read

- Spin 2:

$$\ddot{h}_{ij} - \frac{\dot{a}}{a}\dot{h}_{ij} + \biggl((\frac{k^2}{a^2}) - 2\frac{\ddot{a}}{a}\biggr) h_{ij} = 0 \quad ; \tag{44}$$

- Spin 1:

$$(1+\beta)\dot{\delta u^i} + (1+\beta)(2-3\beta)\frac{\dot{a}}{a}\delta u^i = 0 \quad ; \tag{45}$$

- Spin 0:

$$\begin{aligned} \ddot{h} + 2\frac{\dot{a}}{a}\dot{h} &= 6(\frac{\dot{a}^2}{a^2})(1+\beta)\Delta \quad , && (46)\\ \dot{\Delta} + (1+\beta)(\Theta - \frac{1}{2}\dot{h}) &= 0 \quad , && (47)\\ (1+\beta)\dot{\Theta} + (1+\beta)(2-3\beta)\frac{\dot{a}}{a}\Theta &= \beta(\frac{k^2}{a^2})\Delta \quad , && (48) \end{aligned}$$

where here $\beta = \frac{(5-3\gamma)\alpha+\gamma-1}{3-\gamma-3(1-\gamma)\alpha}$. Hence, the equations (so, the solutions) are similar to those of the standard model, but with a "modified equation of state" in which β replaces α. We can verify that, for $\alpha = -1$, $\beta = -1$, $\alpha = \frac{1}{3}$, $\beta = \frac{1}{3}$, and $\alpha = 0$, $\beta = \frac{\gamma-1}{3-\gamma}$. The results are the same as in the standard model for the false vacuum and radiative phases. During these periods, the rotational modes decrease (or simply does not exist, as in for the false vacuum case), and the density perturbations exist only in the radiative case, but are amplified very moderately.

Let us analyze now the case $\alpha = 0$. For $0 < \gamma < 3$ (so that the weak energy condition is satisfied), $-\frac{1}{3} < \beta < \infty$. For $0 < \gamma < 1$ (growing entropy and older Universe), we have $-\frac{1}{3} < \beta < 0$ (the effective pressure is negative, but the strong energy condition is not violated). We consider now the solutions for the three different modes separately. The scale factor behaves as $a \propto t^{1-\frac{\gamma}{3}}$, or, in terms of the conformal time, $a \propto \eta^{\frac{3-\gamma}{\gamma}}$,

- Spin 2: The solutions can be written as,

$$h_{ij} \propto \eta^{\frac{1}{2}\frac{6-\gamma}{\gamma}} J_\nu(k\eta) \quad , \quad \nu = \frac{6-3\gamma}{2\gamma} \quad . \tag{49}$$

  For large wavelengths, it behaves asymptotically as

$$h_{ij} \propto \eta^{\frac{6-2\gamma}{\gamma}} \quad . \tag{50}$$

  In the interval of interest, the gravitational waves are amplified, but not in a exponential way.

- There are growing rotational perturbations during the material era provided that

$$3\beta - 2 > 0 \Rightarrow \beta > \frac{2}{3} \Rightarrow \gamma > \frac{9}{5} \quad . \tag{51}$$

  However, this leads to an young Universe and a decreasing entropy($\gamma > 1$);

- For the density perturbations, we can note that:

  1. For $\gamma > 1$, the solutions are equivalent to those of the standard model with an equation of state $p = \alpha\rho$, $\alpha > 0$, namely,

$$\Delta \propto \frac{1}{\eta^{\frac{3}{\gamma}}} \int \eta^{\frac{5}{2}} c_\pm J_{\pm\nu}(k\eta) \quad , \quad \nu = \frac{6-3\gamma}{2\gamma} \quad . \tag{52}$$

  2. For $0 < \gamma < 1$ and $\gamma < 0$, the solution are the same as in General Relativity, with $-\frac{1}{3} < \alpha < 0$, which amounts to replacing the Bessel's functions by the modified Bessel's functions (see, e.g. [17]) :

$$\Delta \propto \frac{1}{\eta^{\frac{3}{\gamma}}} \int \eta^{\frac{5}{2}} \Big(c_1 I_\nu(k\eta) + c_2 K_\nu(k\eta) \Big) \quad . \tag{53}$$

  In both cases, the long wavelength limit yields the same asymptotic behaviour

$$\Delta \propto \eta^2 \propto t^{\frac{2}{3}\gamma} \quad . \tag{54}$$

  We can see that the perturbations, in this limit, and with $\gamma > 0$, grow faster than in the Standard Model when the Universe is younger, and slower when the Universe is older. In both cases, we can expect the final amplification of the perturbations to be of the same order. If $\gamma < 0$, the density perturbations are decreasing.

4 Conclusions

Models where the momentum-energy tensor is not conserved *per se* may give reasonable results while coping with some traditional problems posed by the Standard Cosmological Model. Here we have considered two kinds of such theories. The first one is based on a non-trivial interaction between the matter and the scalar field, which can be obtained from the Brans-Dicke theory through conformal transformation. The second one is based on a direct modification of Einstein's theory, via introduction of the parameter γ in Einstein's equations.

The first theory is marred by problems from the Cosmological point of view even in the background solution, since it predicts an Universe younger than the one predicted by the standard model, and decreasing entropy during the matter dominated phase. The second theory, in contrast, gives the opposite result when $0 < \gamma < 1$. For $\gamma < 0$ it predicts an inflationary behaviour during the matter-dominated phase.

Turning to the structure formation problem, we can state the following results.

1. In the realm of the first theory, the gravitational mode has qualitatively the same behaviour than in the standard model. The rotational mode exhibit a growing solution but for a very young Universe. But, contrary to the standard model, there exist density perturbations in the false vacuum case, which, however, have a decreasing behaviour in the long wavelength limit. In the radiative phase, the behaviour is the same than in the standard model, but during the matter dominated phase, the density perturbations grow less rapidly, which is even more serious drawback if we remember that the age of the Universe is also lower than in the standard case;

2. In the second theory, we have the same behaviour that within the standard model if $0 < \gamma < 1$ for gravitational and rotational modes. Rotational modes may exhibit a growing tendency for $\gamma > \frac{9}{5}$ but this leads to a very young Universe and the entropy decreasing during the matter dominated phase. The behaviour of the density perturbations remains unchanged with respect to the standard results during the false vacuum and radiative phases. But during the matter dominated phase there is a fast growing mode (in the long-wavelength limit) if $\gamma > 1$, and a slower one if $\gamma < 1$. The amplification is essentially the same in this phase due to the modifications in the evaluation of the age of the Universe.

The first theory does not lead to acceptable results neither for the background solution, nor at the perturbative level, whereas the second one gives interesting background solutions and, at least, is not disqualified by the perturbation analysis. This can be judged rather encouraging, since as in the standard model, the second theory needs a mechanism for generation and amplification of perturbations in phases prior to the matter dominated era; but, on the other hand, we remember that certain models which give interesting background solutions are

entirely compromised at the perturbative level, as is the case of the models based on string like fluid [5], and which is not the case here.

For $\gamma < 1$, this theory leads to a reasonable entropy production, to an older Universe and to growing modes for density perturbations in the material phase.

Is it possible to extract some observational limit for the parameter γ from the above perturbative analysis? In principle, yes. We could employ for example the observations on the anisotropies of the cosmic microwave background, measured by the COBE satellite. However, in order to do that, we must have a complete scenario at our disposal, including the above mentioned mechanism for the origin and amplification of the perturbations. In our case, this should be done without the help of inflation in order to be coherent with the proposals of the model.

Even without that, we can extract some observational informations. One important observational parameter is the spectral index n, characterizing the spectrum of the perturbations. If $n = 1$ we have the so called Harrison-Zeldovich spectrum: all perturbations, disregarding their wavelength, have the same amplitude when they enter the horizon. The COBE's results indicate a nearly Harrison-Zeldovich spectrum, with $n = 1$ representing possible value.

The spectral index can be evaluated from the two point correlation function. It gives the following result [19, 18],

$$n = 1 + \frac{d\ln\delta^2}{d\ln k} \quad , \tag{55}$$

where δ^2 is proportional to the two point correlation function,

$$|\Delta(\vec{k},t)| \equiv k^3 \int \frac{d^3x}{(2\pi)^3} e^{i\vec{k}.\vec{x}} \langle \Delta(\vec{x},t)\Delta(0,t)\rangle \quad . \tag{56}$$

Using the solutions (52), considering the long wavelength limit with the solution that leads to the Bunch-Davies vacuum (for details, see [19]), we have

$$\delta \propto k^{3-2\nu} \quad , \tag{57}$$

leading to

$$n = 1 + 6\frac{(\gamma - 1)}{\gamma} \quad . \tag{58}$$

The observational limits of n seem to indicate a value very close to $n = 1$. Taking for simplicity (since we are only making a crude estimation) $0.8 < n < 1.2$, we obtain $0.96 < \gamma < 1.03$. Following this simplified analysis, we find that the Universe today must be, from the Theory II, a little older than the one predicted by the standard model, with a very low production of entropy during the matter dominated phase.

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